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MODELS AND MODELLING FOR MANPOWER PLANNING

by

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## MODELS AND MODELLING FOR MANPOWER PLANNING

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### I. Introduction

Growing attention to long-range planning is required of management in most organizations. As businesses grow in size and diversity of operations and try to cope with the challenge of technological change and new markets overseas, they must make increasingly sophisticated, long-range evaluations of investments in products, facilities, and personnel. Governments must build and maintain complex administrative services in fields ranging from social welfare to space exploration. Universities must plan in the face of rising costs, a growing population, and shifts in the kinds and levels of education required by a changing society. The armed services have extended projections of force, research, and equipment requirements for national defense a decade or more into the future.

Some parts of the long-range planning problem have been widely discussed. Other parts have received less attention. Of the relatively neglected areas, one of special importance is manpower planning. Manpower planning includes a specification of the kinds and numbers of men an organization will need to accomplish its profit, growth, or service objectives; a forecast from current personnel inventories of how well it is now set to meet the projected needs; and by a comparison of needs with forecasted supply, the formulation of plans for recruiting, assigning, and developing personnel.

## 2.

This paper explores some issues in manpower planning. We review some of the kinds of approaches that have been tried and look in some detail at two approaches which seem especially promising.

The first of these involves the development of simple stochastic models for aspects of the problem, and the exploitation of these by direct mathematical manipulation. Such an approach requires that the model be an abstraction of the real problem amenable to mathematical and computational tools at hand. However, since there are manifestly a great many complex interactions and effects present in real manpower planning situations, a comprehensive and satisfying model recognizing all of them is probably beyond investigation by known mathematical methods alone. Thus the second of our approaches is to simulate on a computer the histories of a manpower system under different kinds of input, policy, and environmental conditions. In this way the relative importance of the many conceivable factors influencing an organization's manpower status can be identified, and those that are of marginal importance ignored. Eventually it is to be hoped that the mathematical and simulational approaches may be blended, allowing the user to benefit at once from the crispness and conceptual simplicity of the former, while maintaining the comprehensive and evocative character of the latter.

As the final section of the paper argues, both kinds of models will need considerable development before they can satisfactorily reflect all the relevant complexities of personnel policy, organizational structure,

and movements of men from job to job or level to level in an actual organizational system. But to develop and enrich these models we need kinds of data which organizations do not now ordinarily keep. In presenting models in preliminary form, we hope to illustrate the new data requirements and to stimulate wider discussions of ways to put manpower forecasting and planning on a more systematic basis.

## II. The Developing Interest in Manpower Planning

The simplest and oldest approach to manpower planning is perhaps the so-called replacement table method; a list of men or groups of men presently in the system, organized by function and job level, provides a description of the current inventory. The main problem, according to this approach, is to insure that as men quit, retire, or die, suitable candidates will be ready to move into their jobs. Age data lets retirement statistics be predicted with precision and -- coupled with historical information, general actuarial data, and estimates about the future -- also permits estimates of losses for other reasons. Thus, rough estimates can be made of where and when vacancies will occur.

The replacement table method is laborious to carry through by hand computations, and it usually reflects a static rather than a dynamic picture of an organization's structure and needs. The development of other approaches has come since World War II in response to a variety of particular problems.

During World War II, shortages of manpower made planning at the national level a necessity. This was undertaken by the War Manpower Commission which in turn required individual firms to submit regular reports on their current manpower inventory and future requirements; see [1]. For most firms this was their first experience in manpower planning. Shortages of qualified executives during the fifties stimulated the planning of executive development, often starting with training plans

for new college graduates. Thus the principle of developing and stocking manpower for future requirements was elaborated as opposed to the more common practice of responding to specific current needs.

As firms focussed more intently on the planning aspects of the personnel function, the existence of an impact from recruiting, selection, appraisal, assignment and promotion decisions on the quality and quantity of manpower inventories was recognized. While the importance and reality of this relationship was accepted, it was not always possible to structure the relationship in such a way as to assist the firm in decision making. Various techniques were evolved to deal with separate problems. Testing, training, counseling and appraisal have all received considerable attention. Seldom were problems in one or more areas looked at simultaneously; see [2]. This lack of integration is particularly serious when one considers that extreme shortages of professional and managerial personnel are expected to develop before 1970; [3].

Military organizations, on the other hand, have been somewhat more sophisticated about manpower planning. This may be explained partly by the military caste system, which lends itself to formal models, and partly by the special strains which the cold war and a booming economy have placed on the military personnel system. A major problem in the introduction of new weapons systems has been the development of the personnel component to meet the requirements of the total system. This has resulted in several attempts to outline procedures for specifying manpower

requirements and transmitting these data in such a manner and form that the personnel will be trained and available when they are needed; [4]. Defense sub-contractors have also felt a need for better forecasts of requirements at both the firm, [5], and the industry, [6], level.

Several researchers who looked at the supply problem found Markov chains to be a convenient technique for predicting distributions of personnel; [7], [8], [9]. Other techniques proposed were linear programming [10] and network analysis; [11]. The Markovian model was elaborated to show the effects of policy decisions on the personnel system [12], and finally an attempt was made to describe the stocks and flows of manpower in a particular military rating; [13]. This model, while it relied heavily on demographic techniques, went beyond the first simple Markov models to the extent that some policy parameters were included. A computer version was tested and gave reasonably accurate predictions of the behavior of manpower in this rating; [14].

The development of more comprehensive techniques for personnel decision making has emphasized the importance of personnel records and other data as inputs to these decisions; [15]. This is especially true in large organizations where the bulk of the data makes taped computer storage an attractive alternative. In addition, efforts have been made to develop useful ways of describing individual's abilities, [16], and flexible ways of organizing data on the requirements of new and developing jobs; [17].



Policy decisions in the personnel field may have a direct impact on the careers of employers [18], and on the kind of employees who get promoted, their behavior and attitudes; [19]. In turn, the employees' career expectations are related to their attitudes toward the organization and their work; [20]. At a general level, career development in the individual is explained by the development of different needs at different stages of life and consequent exhibition of different behavior; [21]. There is a great deal of work to be done in order to rigorously employ career development concepts in the generation of personnel policies. Current research in this direction is focusing on the identification of career types [22], [23] and patterns [24] and the analysis of the impact of changing skill requirements on career patterns and personnel policies; [25], [26]. Eventually findings about the responses of employees at different career stages to organizational policies and to environmental conditions must be incorporated into models for forecasting the supply of personnel and projecting plans for altering the supply.

### III. Mathematical Models

Any attempt to assess the temporal ebb and flow of inventories of manpower resources through various levels in an organization must reckon with a great many possible influences and interactions, and must necessarily neglect or aggregate many of these in constructing models. Of course inferences drawn from the models will then be tempered accordingly, until the validity or usefulness of the model is established. In developing our present models we shall make a variety of types of simplifying assumptions, in the main attempting to highlight certain interesting or important features of a situation involving personnel flow, while minimizing others deemed to be of less relevance in the particular situation under focus. It should, then, come as no surprise to find that our models cover a wide range of behavioral characteristics for personnel and organizations and that contrasting assumptions are often made to mirror the wide range of possible behavior.

A few words concerning the use of our models. Briefly, we believe that model making is an essential part of the process of understanding a situation as it now is and may become. Furthermore, comprehensible models that may be manipulated are of immense aid in the planning and selection process. Finally, models are important in controlling a process once organized by suggesting system response to organizational action and future environmental changes. Although we have not as yet confronted our models directly with actual data from real organizational experience, we feel

that data analysis conducted with the aid of such models is a profitable enterprise. A study of the causes and importance of deviation of experience from model predictions will help to refine and improve both the model and the understanding of its developer. Among the important results of such analysis is likely to be the collection of new data, meaningful both for understanding and control.

Model 1: Advance in a Rigid Hierarchy

By a hierarchy we mean here an organizational structure like that in the Armed Services or a university faculty, in which recognized ranks occur. Usually higher rank within an organization means higher prestige, salary, authority. However, this ordering often breaks down across career groups or specialty types, e.g., a captain in supply may have less prestige than an infantry captain, a sales vice-president may be paid more than an engineering vice-president, etc. Many business firms are similarly organized, e.g., with a system of plant managers reporting to division managers, who report to vice-presidents, several of whom report to group vice presidents, who in turn report to the president. There are also such hierarchies within plants, or smaller company segments. It is a common pattern that advancement occurs through a career group (e.g., engineering, finance, marketing) up to a certain level, perhaps that of flag rank, or vice president, from which promotions are made to the top positions, e.g., general, admiral, chief of staff, president. Thus a new

entry into the organization must choose a path, or career group, and, if he is ambitious, may wish to select one offering not only present rewards but a good shot at the top slot. What considerations are of importance in his selecting an initial identification with a career, and how may the organization be affected by and possibly control, his behavior? We begin by describing advance in a single career path, later passing to more realistic situations.

A managerial hierarchy consists of a single top-level position, and  $S$  second-level positions. When the top slot is vacated, occupants of the lower slots move up into the vacant top position. We want to investigate the career possibilities of those who enter the hierarchy at the second level.

#### 1. Static Structure - Promotion by Seniority or Experience

First suppose the hierarchical structure never changes, i.e. there are always  $S$  second-level jobs, and one top job. Further, suppose individuals are promoted in order of their seniority, or experience. A new man always starts in the lowest of the second-level ranks (rank  $S$ ), and progresses upward as higher ranking (more senior) individuals either defect from the hierarchy or are themselves promoted. As soon as a vacancy in the structure occurs promotion transmits that vacancy to the lowest ranking position and this vacancy is filled from the outside. The promotions and recruitment changes require negligible time in this model.

As a simple initial probability model for the above process, suppose that individuals in the top position hold that position for times that are independently and exponentially distributed, with mean  $\lambda^{-1}$ . This means that an individual currently holding the top position vacates that position with probability  $\lambda dt$  in a short time interval  $(t, t+dt)$ , regardless of how long he has held that position. Similarly, an individual holding a second level position defects from the hierarchy in  $(t, t+dt)$  with probability  $\mu dt$ . With these simple assumptions we may pose and answer several questions:

- (A) Suppose a man joins the hierarchy at the bottom. If he decides never to defect (and is not fired and does not die), what is the probability that he will reach the top position within a time  $t$  after his entrance?

This problem may be easily solved by adding up the times required to progress one rank at a time up the hierarchical ladder. In order to move from rank  $S$  to  $S-1$  in  $(t, t+dt)$  either the top-level job must be vacated, or one of the  $S-1$  lower jobs vacated, for then everyone below moves up one slot. Such an event occurs with probability  $1-(1-\lambda dt) \times (1-\mu dt)^{S-1} = [\lambda + \mu(S-1)]dt$ , to terms of order  $dt$ . It then follows that the time,  $T_S$ , spent in rank  $S$  has the exponential distribution:

$$P\{T_S \leq t\} = 1 - \exp\{-[\lambda + \mu(S-1)]t\}. \quad (1)$$

Once rank  $S-1$  has been reached the same argument applies again: in order to move up, either the top job or one of the  $S-2$  second-level, but more

senior, positions must be vacated, the probability for which in  $(t, t+dt)$  is  $[\lambda + \mu(S-2)]dt$ , independently of previous events by the memoryless property of the exponential. Hence the time spent in rank  $S-1$ ,  $T_{S-1}$ , is distributed as

$$P\{T_{S-1} \leq t\} = 1 - \exp\{-[\lambda + \mu(S-2)]t\} \quad (2)$$

independently of  $T_S$ . Repeating this process, we see that the time required for the man to reach the top position is

$$T = T_S + T_{S-1} + \dots + T_1, \quad (3)$$

the distribution of which is the convolution of "consecutive" is inferred from (1) and (2). The Laplace transform of the densities of the times in rank are easily written down; for example

$$E(e^{-sT_S}) = \int_0^{\infty} e^{-st} e^{-[\lambda + (S-1)\mu]t} [\lambda + (S-1)\mu] dt = \frac{\lambda + (S-1)\mu}{\lambda + (S-1)\mu + s} \quad (4)$$

so, by independence

$$E(e^{-sT}) = \prod_{j=1}^S E(e^{-sT_j}) = \frac{\lambda + (S-1)\mu}{\lambda + (S-1)\mu + s} \cdot \frac{\lambda + (S-2)\mu}{\lambda + (S-2)\mu + s} \dots \frac{\lambda + \mu}{\lambda + \mu + s} \cdot \frac{\lambda}{\lambda + s} \quad (5)$$

By differentiation we find that the average time to reach the top is

$$E(T) = \frac{1}{\lambda} + \frac{1}{\lambda + \mu} + \frac{1}{\lambda + 2\mu} + \dots + \frac{1}{\lambda + (S-1)\mu} \sim \frac{1}{\mu} \ln(1 + S \frac{\mu}{\lambda}) \quad (6)$$

Thus rate of advance in a hierarchy so organized tends to be rapid at first, slowing down as one rises in the ranks. For the loyal, long-lived

(or unattractive or immobile) individual the waiting time to reach the top from entry tends to go up logarithmically with  $S$ , the number of ranks, rather than linearly.

Another related question is

- (B) Suppose a man joins the hierarchy at the bottom, but is subject to defection and death while in the lower ranks. What is the probability that he will ever serve in the top job?

The answer to this question is easily obtained from expressions (4) and (5). Recognize that by putting  $\mu = s$  in (4) we obtain the probability that a new man reaches rank  $S-1$  without defecting; the same substitution then works for higher ranks. By independence it follows that the probability that the man remains in the hierarchy to the top is

$$E(e^{-\mu T}) = \frac{\lambda + (S-1)\mu}{\lambda + S\mu} \cdot \frac{\lambda + (S-2)\mu}{\lambda + (S-1)\mu} \cdots \frac{\lambda + \mu}{\lambda + 2\mu} \cdot \frac{\lambda}{\lambda + \mu} = \frac{\lambda}{\lambda + S\mu} \quad (7)$$

## 2. Static Structure - Equal Opportunity Selection

Again suppose the organizational structure does not change. When a new man enters the second-level he is considered to be on a par with his peers, having an equal chance at the top-level position each time a vacancy occurs. For this organization we again ask question (A): What is the probability that a tenacious or loyal (or unattractive) man makes it to the top before a time  $t$  after he enters the second level?

This is easily answered by observing that each time an opening occurs at the top there is probability  $\frac{1}{S}$  that any particular individual,  $I$ ,

is selected to fill it. Hence the chance that in  $(t, t+dt)$  I is selected is  $\frac{\lambda}{S} dt$ . It then follows that I reaches the top before time  $t$  with probability

$$1 - \exp[-\lambda t/S] \quad (8)$$

and that

$$E(T_s) = \frac{S}{\lambda} \quad (9)$$

a strictly linear function of  $S$ , to be contrasted with (6). It is noteworthy that in this model the future career possibilities of a man making the second-level do not depend upon the defections of his peers.

Next look at question (B): What is the probability that a man of normal attractiveness reaches the top? This is again obtained by putting  $s=\mu$  in the expectation

$$E(e^{-sT}) = \frac{\lambda/S}{s+\lambda/S}, \quad (10)$$

and gives

$$E(e^{-\mu T}) = \frac{\lambda}{\lambda+\lambda S}, \quad (11)$$

agreeing with the previous expression for the probability of the same event in the seniority model. Note, however, that this equality depends upon the assumption that all individuals have the same defection rate from second-level positions.

This model neglects the important elements of reputation for performance, or loyalty, that may be associated with an individual who



remains for a time in an organization. Thus it may be that after an individual has been in an organization for a time he will be classified as appropriate for promotion, or inappropriate and suitable only for present duties, or possibly for transfer. Our random selection mechanism above should apply only to the eligible class when such a breakdown is made.

Various extensions may be made of the above simple models. First, the  $S$  second-level jobs may be conceived of as being identified with a particular career group, e.g., engineering, or sales; there may then be  $S_1$  second-level jobs for career groups  $i$  ( $i = 1, 2, \dots, c$ ), where  $S_1$  is the number for sales,  $S_2$  for engineering, etc. Then if a top-level vacancy is filled from career group  $i$  with independent probability  $p_i$  ( $p_i > 0$ ,  $p_1 + p_2 + \dots + p_c = 1$ ), and if such vacancies occur at rate  $\lambda$  the previous formulas may be revised to allow the prospects for advancement to be compared across career groups. For example, the average time to reach the top job for a new entry in career group  $i$  is approximately

$$E[T(i)] = \frac{1}{\mu_1} \ln \left( 1 + S_1 \cdot \frac{\mu_1}{\lambda p_1} \right) \quad (12)$$

when promotion by seniority is the rule, and

$$E[T(i)] = \frac{S_1}{\lambda p_1} \quad (13)$$

under equal opportunity selection. For either advancement policy the probability of achieving the top position is given by (7) or (11):

$$E[e^{-\mu_1 T(1)}] = \frac{\lambda p_1}{\lambda p_1 + S_1 \mu_1} \quad (14)$$

Incidentally, it is interesting to remark that, since

$$\frac{1}{\lambda} + \frac{1}{\lambda + \mu} + \dots + \frac{1}{\lambda + (S-1)\mu} < \frac{S}{\lambda}$$

the expected time for a non-moveable second-level employee to reach the top is greater under equal opportunity selection than under seniority, but the probability of reaching the top relatively early is much greater under equal opportunity selection. In any case the expressions (12), (13), and (14) may provide indices of career group attractiveness in a given organization: the fact that (12) or (13) are large, while (14) is low for a group may account for difficulty of recruitment, the necessity of having to accept substandard applicants -- and a subsequent detrimental movement in the index. On the other hand, some career groups (e.g., engineers and scientists) may have no aspirations to other than professional leadership, and the above indices would be of no value for these.

Next, it is worth noting that the restriction to a single top-level position is really unnecessary. If there are  $H$  ( $H = 1, 2, 3, \dots$ ) them, each having a rate of vacating of  $\lambda$ , then we need merely replace  $\lambda$  by  $\lambda H$  in our previous formulas. If the rates of vacating the top jobs differ, we replace  $\lambda$  by the sum of those rates. In combining the rates as suggested there is the implied assumption that the top jobs empty independently, i.e. there is no tendency for a group to leave together, except by pure chance.

On the basis of the above observation it is possible to make a model of a multi-level hierarchy, and the flow through it. Thus, adding a level below the second -- call it the lower level -- of  $L$  jobs, and suppose that defection rate  $\omega$  applies there. Then in order to study the flow from the lower to the second level we may utilize all of the basic formulas first derived, merely replacing  $\lambda$ , the rate of advancement into the next level, by  $\lambda + S\mu$ , or  $H\lambda + S\mu$  if there are  $H$  top jobs, and replacing  $S$  by  $L$  and  $\mu$  by  $\omega$ . For example, the probability that an individual who enters the hierarchy at the bottom of the lower level will reach the second level is, by making the above substitutions in (7) and/or (11),

$$\frac{\lambda + S\mu}{\lambda + S\mu + L\omega} \quad (15)$$

since the probability of passing through the second level without defection (or being fired) is  $\frac{\lambda}{\lambda + S\mu}$ , by (7) or (11), the probability of reaching the top from the extreme bottom is, by independence

$$\frac{\lambda}{\lambda + S\mu} \cdot \frac{\lambda + S\mu}{\lambda + S\mu + L\omega} = \frac{\lambda}{\lambda + S\mu + L\omega} \quad (16)$$

### 3. Two Classes of Recruits, A Fixed Number of Second-Level Jobs

Frequently it becomes necessary to fill second-level slots in a hierarchy with either of two (or more) types of individuals. Thus in the military there are regulars and reserves, and in business organizations differentiations may be made on the basis of training (engineering vs. business), race, or even family connection. When a distinction becomes

obvious, the long-run effect of various promotion policies may not be entirely apparent. The following simple model displays some of the possible dependencies.

Again our hierarchy possesses  $S$  second-level jobs, from which promotions are made to first-level positions at rate  $\lambda$ . Distinguish two classes of individuals in the second-level positions: type A, and type B; refer to  $A(t)$ , the number of type A individuals in the second-level jobs as the state of the system at time  $t$ . For the present we assume that all  $S$  jobs are filled, so  $S-A(t)$  is the number of type B individuals present at time  $t$ . We shall model  $A(t)$  by a simple birth-and-death stochastic process (see Feller [32], Chap. XVII). The following additional parameters will be introduced:

$\alpha_n, (\beta_n)$  = defection rate (quit rate, or discharge rate) of Type A, (B) when the state of the system is  $n$ ;

$a_n, (1-a_n)$  = probability that a Type A, (B) is recruited to fill a second-level vacancy, given that system state is  $n$  just before recruitment;

$\pi_n, (1-\pi_n)$  = probability that a Type A, (B) individual is promoted to a first-level vacancy, given that system state is  $n$  just before the vacancy occurs.

Assuming independence of the above events, and utilizing these parameters, we may describe the system state as a birth and death process (see Feller [32], Chap. XVII): in any time interval of duration  $dt$  the transition probabilities are, to terms of order  $dt$ ,

<u>Transition</u>	<u>Probability</u>
n to n+1	$\lambda_n dt = [\lambda(1-\pi_n) + \beta_n] a_n dt$
n to n-1	$\mu_n dt = [\lambda\pi_n + \alpha_n](1-a_n)dt$

By way of explanation, a transition from n to n+1 occurs if either

1) a promotion is made, the individual is of Type B, and an A is recruited for replacement, or ii) a B defects and an A is recruited to replace him.

The sum of the probabilities of the events i) and ii) are recognizable as

$\lambda_n dt$ ; by a similar argument we obtain  $\mu_n dt$ .

The stationary or long-run probability that system state is n may be expressed (see Cox and Smith [33], p. 43) as

$$P_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} P_0 \quad (17)$$

where  $P_0$  is determined by the condition that

$$\sum_{n=0}^S P_n = 1 \quad (18)$$

The expression (17) thus describes the inventory of Type A, and hence Type B, individuals in second-level jobs in the long run; the relative numbers of the two types is of course determined by the various parameters described above. Of course the inventory probabilities  $\{P_n, n = 0, 1, \dots, S\}$  also influence the type of individual occupying the top-level jobs. Indeed, one can easily see that the long-run probability with which top-level positions are filled with Type A individuals is

$$\sum_{n=0}^S \pi_n P_n;$$

since the tenure distribution of Type A is here assumed no different from that of a Type B -- an easily removed simplification -- the latter expression also gives the probability that a Type A occupies the top job in the long run.

Some specific examples may be of interest. In these the parametric dependence upon system state will be specialized.

(A) Let

$\alpha_n = n\alpha$ ,  $\beta_n = (S-n)\beta$ , so defections of either type are proportional to the numbers of each present;

$a_n = a$ , so the recruitment rate of either type does not depend upon the second-level occupancy, but only upon the supply;

$\pi_n = \frac{n}{S}$ , so the probability that an A is actually promoted is simply proportional to the number present -- this is equal opportunity promotion.

Then, writing out the expression for  $P_n$ , we have

$$P_n = \left\{ \frac{[\frac{\lambda}{S} + \beta] S a [(\frac{\lambda}{S} + \beta)(S-1)a] \dots [(\frac{\lambda}{S} + \beta)(S-n+1)a]}{[\frac{\lambda}{S} + \alpha](1-a) [\frac{\lambda}{S} + \alpha] 2(1-a) \dots [(\frac{\lambda}{S} + \alpha)n(1-a)]} P_0 \right\} \quad (19)$$

which, after simplification, becomes

$$P_n = \frac{S!}{(S-n)!n!} \left[ \frac{(\frac{\lambda}{S} + \beta) a}{(\frac{\lambda}{S} + \alpha)(1-a)} \right]^n P_0 \quad n = 0, 1, 2, \dots, S. \quad (20)$$

Finally, summing the binomial series, we get the explicit formula

$$P_n = \frac{S!}{n!(S-n)!} p^n (1-p)^{S-n}, \quad (21)$$

where

$$p = \frac{(\lambda + \beta S)a}{(\lambda + \beta S)a + (\lambda + \alpha S)(1-a)}. \quad (22)$$

Thus system state has the binomial distribution. Thus it follows that the top job is occupied by an A with probability

$$\sum_{n=0}^S \pi_n P_n = \sum_{n=0}^S \frac{n}{S} P_n = \frac{1}{S} \cdot Sp = p. \quad (23)$$

Another piece of worthwhile information to be culled from (21) is the long-run average rate of A and B defection. For A's, this is

$$\sum_{n=0}^S \alpha_n P_n = \sum_{n=0}^S n \alpha P_n = \alpha Sp \quad (24)$$

and a comparable result holds for B's.

(B) Let all of the parametric assumptions agree with those of A, except that

$$\pi_n = \begin{cases} 1 & \text{if } n = 1, 2, \dots, S \\ 0 & \text{if } n = 0 \end{cases} \quad (25)$$

This reflects an over-riding propensity to promote A's if any are present.

Upon substituting into (17) we find for the stationary probabilities

$$P_n = \frac{S!}{n!(S-n)!} \left(\frac{a}{1-a}\right)^n \frac{\left(\frac{\lambda}{S} + \beta\right) \beta^{n-1}}{\left(\frac{\lambda}{1} + \alpha\right)\left(\frac{\lambda}{2} + \alpha\right) \dots \left(\frac{\lambda}{n} + \alpha\right)} P_0 \quad (26)$$

where again  $P_0$  is determined by the normalization condition (18).

Unfortunately, the expression (26) cannot be summed in convenient closed form. However, there would be little trouble in computing numerically such measures as the probability that an A (or B) occupies the top job, and the long-run average rate of defection of A's and B's; the former probability is

$$\sum_{n=0}^S \pi_n P_n = \sum_{n=1}^S P_n,$$

and the probability that an A defects is

$$\sum_{n=0}^S n \alpha P_n.$$

## Model 2: Advance in a Flexible Hierarchy

The present model is an attempt to model the advance of individuals through a rank structure in which no constraints are placed on the number of individuals simultaneously occupying a given rank. In this respect individuals compete only against themselves for progress in the hierarchy; there is need only for a recognized increase in competence in order to advance, and not a vacant slot. Such may be the situation of one entering a sparsely populated field. Possibly this model applies to the electronic or nuclear technician in the armed services.



In this model

- (1) Ranks are labelled in increasing order as 1, 2, 3, ..., R; rank 1 is lower than rank 2, etc., rank R is the highest;
- (2) An individual who reaches rank  $r$  ( $1 \leq r \leq R$ ) is endowed with a time in rank,  $T_r$ : a random variable with distribution

$$P[T_r \leq t] = F_r(t), \quad (27)$$

independent of time spent in previous ranks, and of the other individuals transitioning through the ranks at the same time.

(3) Each time an individual enters a new rank, a random choice is made: with probability  $\alpha_r$  the individual remains in service until advanced to rank  $r + 1$ , and with probability  $1 - \alpha_r$  he eventually leaves the rank structure (e.g. military service), in which case we say he enters rank 0, where he remains permanently thereafter. Remark that while the alternatives appear to be "either up or out", there is no difficulty in accomodating those individuals who remain at a certain intermediate rank level and spend the remainder of their career there, finally transitioning to rank 0 at retirement.

(4) Given that an individual remains in service until he is advanced, his time in rank is the random variable  $A_r$ , with distribution function

$$P[A_r \leq t] = H_r(t). \quad (28)$$

Given that the individual defects from rank  $r$  of the rank structure his time in rank is  $D_r$ , and

$$P[D_r \leq t] = L_r(t). \quad (29)$$

Thus unconditional time in rank,  $T_r$ , has distribution

$$F_r(t) = \alpha_r H_r(t) + (1-\alpha_r) L_r(t). \quad (30)$$

Detailed hypotheses may be made about the distribution and probabilities introduced; this will be done later.

The previous assumptions are sufficient to specify probabilities for the rank, or state, of an individual at time  $t$  after his entrance into the rank structure. Let  $R(t)$  denote an individual's rank at time  $t$ , given that  $R(0) = 1$ , i.e. that the individual entered rank  $r = 1$  initially. Then

$$P_{1r}(t) = P[R(t) = r | R(0) = 1] = \prod_{j=1}^{r-1} \alpha_j \star \prod_{j=1}^{r-1} H_j(t) [1-F_r(t)], \quad r \geq 1. \quad (31)$$

The symbol  $\star$  denotes convolution. (31) merely expresses the fact that in order for an individual to be in rank  $r$  at time  $t$ , he must (i) have steadily advanced through the preceding  $r-1$  ranks, an event of probability  $\prod_{j=1}^{r-1} \alpha_j$ , (ii) have spent a total time  $A_1 + A_2 + \dots + A_{r-1}$  in reaching state  $r$ , with

$$\star \prod_{j=1}^{r-1} H_j(t)$$

representing the distribution of that time, and (iii) still be in state  $r$  at time  $t$ , neither having advanced nor left the rank structure.

Various other system properties are easily calculated. For example, the probability that an individual stays in the rank structure

until rank  $r$  ( $1 \leq r \leq R$ ) is reached, and then defects is

$$P[\text{defection at rank } r] = \prod_{j=1}^{r-1} \alpha_j (1 - \alpha_r) . \quad (32)$$

EXAMPLE. We now illustrate some of the previous ideas. Suppose that the normal duration of time in rank is the random variable  $S_r$ , with distribution  $U_r(t)$ ; and that, independently, the probability of defection in any time interval  $(t, t + dt)$  is  $\lambda dt + o(dt)$ ; i.e. a time-homogeneous Poisson process describes defections. Then, given that an individual is entering rank  $r$ , his probability of avoiding defection while in that rank is

$$\begin{aligned} \int_0^{\infty} e^{-\lambda t} dU_r(t) &= U_r(\lambda) , \\ &= \alpha_r \quad 1 \leq r \leq R-1 , \end{aligned} \quad (33)$$

the Laplace-Stieltjes transform of the distribution  $U$ .

EXAMPLE. If  $S_r$  were constant (each rank period the same duration,  $\tau$ ) then

$$U_r(t) = \begin{cases} 0, & 0 \leq t < \tau \\ 1, & \tau \leq t \end{cases} \quad (34)$$

and

$$\alpha_r = e^{-\lambda \tau} . \quad (35)$$

The probability distribution  $H_I(t)$  is seen to be given by

$$H_I(t) = \frac{\int_0^t e^{-\lambda x} dU_I(x)}{\hat{U}_I(\lambda)} \quad (36)$$

with

$$\hat{U}_r(s) = \frac{\hat{U}_r(s+\lambda)}{\hat{U}_r(\lambda)} \quad (37)$$

and

$$L_r(t) = \frac{\int_0^t [1 - U_r(x)] e^{-\lambda x} \lambda dx}{1 - \hat{U}_r(\lambda)} \quad (38)$$

with

$$\hat{L}_r(s) = \frac{\lambda}{\lambda+s} \frac{1 - \hat{U}_r(\lambda+s)}{1 - \hat{U}_r(\lambda)} . \quad (39)$$

Then from (2.4), using (2.7), (2.10), and (2.12) we find

$$\begin{aligned} F_r(t) &= \int_0^t e^{-\lambda x} d U_r(x) + \int_0^t [1 - U_r(x)] e^{-\lambda x} \lambda dx. \\ &= 1 - e^{-\lambda t} [1 - U_r(t)] \end{aligned} \quad (40)$$

and

$$\hat{F}_r(s) = \hat{U}_r(s+\lambda) + \frac{\lambda}{\lambda+s} [1 - \hat{U}_r(s+\lambda)] . \quad (41)$$

Further, it is easy to write down the Laplace transform

$$\hat{p}_{1r}(s) = \int_0^\infty e^{-st} \hat{p}_{1r}(t) dt = \prod_{j=1}^{r-1} \alpha_j \hat{H}_j(s) \cdot \frac{1 - \hat{F}_r(s)}{s} . \quad (42)$$

If the conditions leading to (34) and (35) are fulfilled we get

$$\hat{p}_{1r}(s) = [e^{-(s+\lambda)\tau}]^{r-1} \left\{ \frac{1 - e^{-(s+\lambda)\tau}}{\lambda+s} \right\} \quad 1 \leq r \leq R . \quad (43)$$

In order to describe the probability distribution of those in the 0, or defected, rank at time  $t$ , note that

$$\sum_{r=1}^R P_{1r}(t) + P_{10}(t) = 1, \quad (44)$$

so, taking transforms and summing, we obtain from (2.17)

$$p_{10}(s) = \int_0^{\infty} e^{-st} p_{10}(t) dt = \frac{\lambda}{s(\lambda+s)} + \frac{1}{\lambda+s} e^{-(\lambda+s)\tau R}. \quad (45)$$

The assumption of constant defection rate,  $\lambda$ , mentioned above is subject to criticism, and is by no means essential to the model. Defection rate may increase with rank, up to a point, indicating that individuals may become increasingly attractive to other organizations as they advance, due to increased experience and training. If we make defection a random (Poisson-type) process, with rate  $\lambda_r$  dependent upon rank we have only to alter matters slightly by putting  $\lambda_r$  for  $\lambda$  in any formula for a function or quantity with subscript  $r$ .

The foregoing development has emphasized the time-dependent behavior of single individuals in a hierarchy. Going further, of considerable interest to an organization that requires staffing is a projection of the total number of individuals in various ranks for some future date, given that definite policies of recruitment and advancement are followed, and taking account of likely defections. The preceding section has supplied a first model for individual progress or defection; we now want to

combine this class of models with expressions describing the manner in which individuals enter the organizational rank structure. In what follows some special examples are given; they are illustrative only and may be extended to handle other interesting cases as they arise.

Model 2a. Suppose recruitment to rank 1 of the organization occurs according to a time-homogeneous Poisson process with rate  $\nu$ . Then, in time  $T$ , the number of individuals,  $N$ , who have entered the rank structure has expectation  $\nu T$ . By a familiar property of the stationary Poisson process, conditional on  $N = n$ , the time of entry of each of the  $n$  individuals is uniformly and independently distributed over  $(0, T)$ , so the probability that an individual who enters at some time between 0 and  $T$  is at rank  $r$  at time  $T$  is

$$\bar{P}_{1r}(T) = \int_0^T P_{1r}(t) \frac{dt}{T}, \quad r = 0, 1, 2, \dots, R. \quad (46)$$

Given  $N = n$ , the number of individuals in the various ranks has the multinomial distribution, so the joint generating function of the number of individuals in each rank is ( $z_i$  is the g.f. variable for rank  $i$ ).

$$g(z_0, z_1, \dots, z_R | N = n) = [z_0 \bar{P}_{10}(T) + z_1 \bar{P}_{11}(T) + \dots + z_R \bar{P}_{1R}(T)]^n. \quad (47)$$

Finally, the joint generating function of the number of individuals who have entered in the time interval  $(0, T)$  that are in each rank at time  $T$  is

29).

$$\begin{aligned}
 g(z_0, z_1, \dots, z_R) &= \sum_{n=0}^{\infty} g(z_0, z_1, z_R | N = n) e^{-\nu T} \frac{(\nu T)^n}{n!} \\
 &= \exp \left\{ -\nu T + \nu T [z_0 \bar{P}_{10}(T) + z_1 \bar{P}_{11}(T) + \dots + z_R \bar{P}_{1R}(T)] \right\}.
 \end{aligned} \tag{48}$$

Expression (48) simply states that the number of individuals in each rank is independently and Poisson distributed. By differentiation of the generating function we find

$$E[N_r(T)] = \nu T \bar{P}_{1r}(T) = \nu \int_0^T P_{1r}(t) dt. \tag{49}$$

From (31) or (42) it follows directly that

$$\lim_{T \rightarrow \infty} E[N_r(T)] = \nu \prod_{j=1}^{r-1} \alpha_j E(T_r) \quad r = 1, 2, \dots, R. \tag{50}$$

Now an index of the long-run distribution of the relative numbers of individuals in the various ranks (not counting  $r = 0$ ) is

$$f_r = \frac{\lim_{T \rightarrow \infty} E[N_r(T)]}{\sum_{r=1}^R \lim_{T \rightarrow \infty} E[N_r(T)]} = \frac{\prod_{j=1}^{r-1} \alpha_j E(T_r)}{\sum_{r=1}^R \prod_{j=1}^{r-1} \alpha_j E(T_r)}. \tag{51}$$

EXAMPLE. Under the conditions leading to expression (43), i.e. constant time in rank and rank-independent defection rate,

$$f_r = \frac{(e^{-\lambda \tau})^{r-1} (1 - e^{-\lambda \tau})}{1 - e^{-\lambda \tau R}} \quad r = 1, 2, \dots, R. \tag{52}$$

Operationally, formulas (51), or (52), have the following meaning:

consider a large number of organizations, each with the same rank structure and defection rates, and average the number of individuals in the rank  $r$  ( $r = 1, 2, \dots, R$ ) at time  $T$ ; then the ratio of the average number in rank  $r$  to the total number remaining in the organization is approximately  $f_r$ .

For many organizations, perhaps, for example, some branch of the armed services, the rate of entry,  $v$ , can be taken to be large. In this case  $N_r(T)$  is approximately Gaussian, with mean and variance equal to  $vT \bar{P}_{1r}(T)$ . Then consider the rank structure of a single organization at time  $T$ , as indicated by

$$\varphi_r(T, v) = \frac{N_r(T)}{\sum_{r=1}^R N_r(T)} . \quad (53)$$

$\varphi_r(T, v)$  is, of course, a random variable, but as  $v \rightarrow \infty$  it tends in probability to the ratio

$$\varphi_r(T, v) = \frac{\bar{P}_{1r}(T)}{\sum_{r=1}^R \bar{P}_{1r}(T)} . \quad (54)$$

Now as  $T \rightarrow \infty$  the latter ratio approaches  $f_r$ , so

$$\lim_{T \rightarrow \infty} \lim_{v \rightarrow \infty} \varphi_r(T, v) = f_r \quad (55)$$

Consequently, under the preceding assumptions a relative rank distribution, as described by  $f_r$ , is eventually attained. Such a relative distribution



helps to indicate the prospective rank occupancy of the organization, provided internal and external conditions remain unchanged.

In above discussion we have ignored rank occupancy prior to the initial moment,  $T = 0$ , so our conclusions apply only to an organization beginning after  $T = 0$  and experiencing rather constant recruitment. Actually, many organizations experience occasional changes in the rate of recruitment. We suggest models mirroring such effects.

Model 2b. Suppose an organization begins making additions to work force in the time interval  $(0, T')$ ; additions are stationary Poisson of rate  $v'$ . At  $T'$  the rate changes to  $v$ , which prevails until time  $T$ . Let us examine the rank occupancy at time  $T$ . Such a model might describe the situation in the enlisted grades of the armed forces if compulsory military service were terminated at time  $T'$ .

Again one can argue conditionally, using convenient Poisson process properties. The probability that an individual who enters the organization in the initial period,  $(0, T')$ , is at rank  $r$  at time  $T$  is

$$\bar{P}_{1r}(T; 0, T') = \int_0^{T'} P_{1r}(T-t) \frac{dt}{T'} = \frac{1}{T'} \int_{T-T'}^T P_{1r}(t) dt. \quad (56)$$

The joint generating function of the number of individuals who have entered in  $(0, T')$  who are in the various ranks at  $T$  is

$$g(z_0, z_1, \dots, z_R | T; 0, T') = \exp \left\{ -v'T' + v'T' [z_0 \bar{P}_{10}(T; 0, T') + \dots + z_R \bar{P}_{1R}(T; 0, T')] \right\} \quad (57)$$

Similarly, the probability that an individual who entered the organization in  $(T', T)$  is at rank  $r$  at time  $T$  is

$$\bar{P}_{1r}(T; T', T) = \frac{1}{T-T'} \int_0^{T-T'} P_{1r}(t) dt. \quad (58)$$

The joint generating function of the number entering in  $(T', T)$  is

$$g(z_0, z_1, \dots, z_R | T; T', T) = \exp \left\{ -\nu(T-T') + \nu(T-T') [z_0 \bar{P}_{10}(T; T', T) + \dots + z_R \bar{P}_{1R}(T; T', T)] \right\}. \quad (59)$$

By independence, the joint generating function for all individuals who have entered in  $(0, T)$  is the product of (57) and (59) which can be written as

$$g(z_0, z_1, \dots, z_R | T; 0, T) = \exp \left\{ -[\nu' T' + \nu(T-T')] + \sum_{i=0}^R z_i \left[ \nu' \int_0^{T'} P_{1i}(T-t) dt + \nu \int_{T'}^T P_{1i}(T-t) dt \right] \right\}. \quad (60)$$

Consequently it again follows that  $N_r(T)$ , the number of individuals in rank  $r$  at  $T$ , is Poisson distributed independently of those in other ranks, and

$$E[N_r(T)] = \nu' \int_0^{T'} P_{1r}(T-t) dt + \nu \int_{T'}^T P_{1r}(T-t) dt. \quad (61)$$

Model 2c. The next variation of the previous model is that in which entry rates change at regular intervals; the number of entries in successive

time periods are, however, still independently distributed Poisson-wise. Suppose the time  $(0, T)$  is split into  $n$  equal subperiods of length  $\Delta = \frac{T}{n}$ . Then the probability that an individual who enters during the  $j^{\text{th}}$  time period, i.e. at some time  $t$ , where  $(j-1)\Delta < t \leq j\Delta$ , is in rank  $r$  at time  $T$ , is

$$P_{1r}(t; j) = \int_{(j-1)\Delta}^{j\Delta} P_{1r}(T-t) \frac{dt}{\Delta}. \quad (62)$$

Now if  $v_j$  is the rate at which individuals enter the system during the  $j^{\text{th}}$  time period, we have for the joint generating function of the number of individuals in the various ranks at time  $t$

$$g(z_0, z_1, \dots, z_R | T; \Delta) = \exp \left\{ \sum_{j=1}^{T/\Delta} v_j \Delta + \sum_{r=0}^R z_r \sum_{j=1}^{T/\Delta} v_j \Delta \int_{(j-1)\Delta}^{j\Delta} P_{1r}(T-t) \frac{dt}{\Delta} \right\}. \quad (63)$$

Now if  $v(t)$  is a continuous function over  $(0, T)$  and  $v_j$  is some value intermediate between the supremum and infimum of  $v(t)$  for  $(j-1)\Delta \leq t \leq j\Delta$ , and if  $P_{1r}(t)$  is continuous then by letting the number of subperiods increase, so  $n \rightarrow \infty$ , or, equivalently,  $\Delta \rightarrow 0$ , the internal sums in the above tend to Riemann integrals, and

$$g(z_0, z_1, \dots, z_R | T; 0) = \exp \left\{ - \int_0^T v(t) dt + \sum_{r=0}^R z_r \int_0^T v(t) P_{1r}(T-t) dt \right\} \quad (64)$$

EXAMPLE. Suppose entry rate is

$$v(t) = w e^{\theta t}, \quad w > 0 \quad (65)$$

so entries are increasing ( $\theta > 0$ ) or decreasing ( $\theta < 0$ ) exponentially. Now

$$\int_0^T v(t) dt = v \int_0^T e^{\theta t} dt = \frac{v}{\theta} (e^{\theta T} - 1)$$

and

$$\begin{aligned} \int_0^T v(t) P_{1r}(T-t) dt &= v \int_0^T e^{\theta t} P_{1r}(T-t) dt = v e^{\theta T} \int_0^T e^{-\theta t} P_{1r}(t) dt \\ &\sim v e^{\theta T} P_{1r}(\theta) \quad \text{as } T \rightarrow \infty, \end{aligned} \quad (66)$$

for  $\theta$  greater than the abscissa of convergence of the Laplace transform.

This shows that  $N_r(T)$  remains Poisson, and that

$$E[N_r(T)] = v e^{\theta T} \int_0^T e^{-\theta t} P_{1r}(t) dt \sim v e^{\theta T} P_{1r}(\theta). \quad (67)$$

Thus it follows that

$$f_r = \frac{\lim_{T \rightarrow \infty} E[N_r(T)]}{\sum_{r=1}^R \lim_{T \rightarrow \infty} E[N_r(T)]} = \frac{\hat{P}_{1r}(\theta)}{\sum_{r=1}^R \hat{P}_{1r}(\theta)}. \quad (68)$$

EXAMPLE. Substitute into the expression (68) the transform (43); the result is

$$f_r = \frac{[e^{-(\theta+\lambda)\tau}]^{r-1} [1 - e^{-(\theta+\lambda)\tau}]}{1 - e^{-(\theta+\lambda)\tau R}}, \quad r = 1, 2, \dots, R. \quad (69)$$

Notice that the larger  $\theta$  the more the distribution is shifted towards the lower ranks; this makes good sense because a large  $\theta$  means that the number

of entries is increasing, and of course all of these start at the bottom. However, the "shape" of the fraction,  $f_r$ , remains of geometric form.

For a single organization it is again interesting to look at

$$\varphi_r = \frac{N_r(T)}{\sum_{r=1}^R N_r(T)} \quad (70)$$

This random variable tends to a constant as  $v \rightarrow \infty$ ; as  $T \rightarrow \infty$  as well,  $\varphi_r$  approaches  $f_r$ .

Observe that the entry rate (65) provides a convenient building block for modelling other, more complex and realistic, patterns. For example,  $v(t) = v_0(1 - e^{-\theta t})$  represents a rate that increases, but eventually levels off at  $v_0$ . One can also contemplate introducing complex values of  $\theta$ , which would represent cyclic behavior. Such possibilities will be left for future development.

Finally, let us re-examine our model for individual advance, suggesting an **alternative**. The assumption that, for any given individual, his sequence of times in rank  $(T_1, T_2, \dots, T_r, \dots, T_R)$  is one of independently identically distributed random variables is likely to be a considerable oversimplification. Suppose promotion is determined by some relatively objective criterion, such as passing a set of examinations. Because of individual differences we would expect that, other things being equal, a positive association would exist between times in successive ranks for

different individuals: e.g., if  $T_1$  is "short" for an individual, more than likely  $T_2$  will be "short" as well, etc. One simple and illustrative, not necessarily realistic, example is the following:

Model 2d. Suppose that individual differences are represented by a parameter  $D$  that is Gamma distributed over the population of applicants. Thus the density of  $D$  is

$$f_D(x) = e^{-ax} \frac{(ax)^{k-1}}{\Gamma(k)} a \quad \begin{matrix} a > 0, & k > 1 \\ x \geq 0 \end{matrix} \quad (71)$$

with

$$E(D) = m = \frac{k}{a}, \quad \text{and} \quad \text{Var}(D) = \frac{k}{a^2} \quad (72)$$

so

$$k = \frac{m^2}{\sigma^2} \quad \text{and} \quad a = \frac{m}{\sigma^2}.$$

Let the individual's duration of time in rank (no defection) be exponential,

$$H_r(t; D) = 1 - e^{-\lambda_r t} \quad (73)$$

where  $\lambda_r^{-1}$ , the mean time in rank, is dependent upon the value of  $D$ ; in fact, take

$$\lambda_r = D. \quad (74)$$

Thus, individual differences are, operationally, individual advance or up-transition, rates. The Gamma distribution for rates recommends itself

because it is the natural conjugate distribution for Bayesian purposes, and because it has sufficient flexibility to be useful here. It does not, however, combine analytically in a way that is always useful; other models will be required, and are presented subsequently.

This simple model has some of the qualitative properties desired. First, times in tank are correlated, although conditional on the value of  $D$  they are independent:

$$E(T_1 T_j | D) = E(T_1 | D) E(T_2 | D) = \frac{1}{\lambda_1} \frac{1}{\lambda_2} = \frac{1}{D^2} \quad (75)$$

by assumption. However, removing the condition by use of (71) there results (assuming in all that is done that integrals exist),

$$E(T_1 T_j) = \int_0^{\infty} x^{-2} \frac{e^{-ax} (ax)^{k-1}}{\Gamma(k)} a \, dx = \frac{a^2}{(k-1)(k-2)}. \quad (76)$$

Since

$$E(T_1) = \int_0^{\infty} x^{-1} \frac{e^{-ax} (ax)^{k-1}}{\Gamma(k)} a \, dx = \frac{a}{k-1} \quad (77)$$

covariance  $(T_1, T_j) = E(T_1 T_j) - E(T_1) E(T_j)$

$$= \frac{a^2}{(k-1)(k-2)} - \frac{a^2}{(k-1)^2} = \frac{\frac{m^2}{\sigma^2}}{\left(\frac{m^2}{\sigma^2} - 1\right)^2 \left(\frac{m^2}{\sigma^2} - 2\right)}. \quad (78)$$

Since

$$E(T_1^2) = \int_0^{\infty} 2x^{-2} \frac{e^{-ax} (ax)^{k-1}}{\Gamma(k)} a \, dx = \frac{2a^2}{(k-1)(k-2)}. \quad (79)$$

Variance

$$(T_1) = \frac{2a^2}{(k-1)(k-2)} - \frac{a^2}{(k-1)^2} = \frac{a^2 k}{(k-1)^2 (k-2)}, \quad (80)$$

so

$$\rho_{1j} = \text{correlation}(T_1, T_j) = \frac{1}{K} = \frac{\frac{\sigma^2}{m}}{\frac{\sigma^2}{m}} < \frac{1}{2}, \quad (81)$$

and the bigger the variation between individuals, relative to mean rate, the greater is the correlation between within-rank durations for individuals.

Next compute the expected time to reach rank  $r$ , starting from rank 0. Given  $D$ ,

$$E(T_1 + T_2 + \dots + T_{r-1} | D) = \frac{r-1}{D}, \quad (82)$$

so

$$E(T_1 + T_2 + \dots + T_{r-1}) = (r-1) \frac{\frac{a}{k-1}}{\frac{\frac{m}{2} - 1}{\sigma}} = (r-1) \frac{\frac{\sigma^2}{2}}{\frac{\frac{m}{2} - 1}{\sigma}} = \frac{(r-1)m}{\frac{m}{2} - \sigma^2}, \quad (83)$$

and the effect of introducing variability into advance rate according to the present model is to increase the expected time for an individual to reach rank  $r$ . Parenthetically, Cauchy's inequality applied to (82) after the condition is removed shows that, whatever the population from which entrants are drawn, then, if  $E(D)$  is fixed, increase in the variability of  $D$  tends to increase the time to reach rank  $r$ . This is also apparent from (83).

Given the previous assumptions there is no difficulty in finding the probability that an individual is at rank  $r$  at time  $t$  after entry. Given  $D$  it is the probability that the individual has made exactly  $r-1$  jumps, and the latter is the Poisson expression



$$P_{1r}(t;D) = e^{-Dt} \frac{(Dt)^{r-1}}{(r-1)!}, \quad r = 1, 2, \dots \quad (84)$$

so if the condition on D is removed via (71),

$$\begin{aligned} P_{1r}(t) &= \int_0^{\infty} e^{-xt} \frac{(xt)^{r-1}}{(r-1)!} \frac{e^{-ax}(ax)^{k-1}}{(k-1)!} a \, dx \\ &= \frac{(r+k-2)!}{(r-1)!(k-1)!} \left(\frac{t}{t+a}\right)^{r-1} \left(\frac{a}{t+a}\right)^k. \end{aligned} \quad (85)$$

Needless to say, there are other mechanisms for generating an observed correlation between time in rank for individuals. Among them is a sort of "halo effect", in which a quick promotion may engender another for reasons of visibility, personal encouragement and consequent momentum, etc. Treatment of such processes is in progress.

#### Comments

The foregoing models are merely a small selection of those that may be constructed for describing personnel flow and stocks. Many others may be developed emphasizing various features of the problems, and suggesting new and meaningful questions and investigations.

Having descriptive models of the sort introduced above, a next step is to become explicitly normative or decision-theoretic about the personnel process. Thus one may influence costs and objectives by altering system parameters such as S, the number of second-level jobs, or v, the rate of intake of individuals. Of course there may well be reactions to

organizational re-design, e.g. the voluntary defection rate from second-level jobs may decrease if the number of these is made smaller, implying a greater chance of further advancement. Perhaps with the aid of explicit models and a certain amount of experimentation, inferences about the quantitative behavior of individuals in organizations may be strengthened, and trends spotted and initiated; possibly such control may be exerted by simply publicizing the implications of the organization's policy and recent history.

#### IV. Computer Simulation

Implicit in any real process of personnel movements through a hierarchy, there are sets of decision rules. These decision rules may be classified as either structural, policy, or individual rules. The structural category refers to rules which modify and control the structure of the organization. This would include rules for creating or abolishing positions, ranks or departments. It would also include rules about flows of information, influence and products. This category deals only with the formal organization and ignores information about the differences which may exist between the individuals who occupy the positions. The policy category refers to those rules which are used to select and allocate personnel and must therefore take individual differences into consideration. The individual category includes those rules which decide whether an individual will join, remain in, or leave the organization. They may also deal with an individual's decision to change one of his attributes (e.g. productivity).

In order to model personnel flows, and to manipulate the models either mathematically or by computer simulation, it is necessary to make the decision rules explicit, simple, and operational. Many different sets of rules may be compared, but, departing from the model types of the last section, perhaps the following will be of interest:

Rigid vs. flexible structure. Under the rigid structure the number of ranks and the number of positions in each rank is fixed. In each period sufficient personnel are moved into each rank to satisfy these

requirements. With a flexible structure the number of ranks is fixed but the number of positions in each rank is simply the outcome of the processes of hiring, promotion and attrition. Sufficient personnel are hired each period to keep the total manpower at a fixed level. Mathematical methods of the type illustrated earlier will serve to answer some questions that naturally arise. However, in order to study the sensitivity of results to basic assumptions (e.g. independence, distributional form, the relevance of stationary or long-run solutions), to observe the actual time evolution of the processes involved, and to synthesize large-scale models, computer simulation suggests itself.

Seniority vs. merit selection policy. The heading adequately states the distinction between the policies; we have looked at simple models in Part III. However, in order to provide the individuals with differentiated organizational experience in addition to their variation in attributes, the selection policies are made stochastic. Thus, increased seniority or merit simply increases the probability of promotion.

Uninformed vs. informed individuals. For the uninformed individual the decision to leave is independent of his attributes, experience and environment. The probability of leaving is constant across ranks and individuals. For the informed individual the probability of leaving increases as the actual time-in-rank exceeds the expected time-in-rank. The expected time is a function of the rate of promotion of the individual and the rate of promotion of comparable individuals.

It should be noted that for the set of decision rules combining flexible structure and uninformed individuals with either seniority or merit based promotion, this model can be set up as a Markov chain. One of the advantages of using this technique is the ease with which information can be extracted from the model. For instance, expected time in each state and in the system, variance of these quantities, expected number in a given state and its variance, probability of reaching any given state and the expected time to reach that state and its variance, expected number of states occupied by an individual and its variance, mean and variance of the number of changes of state, are all available from simple calculations. Our mathematical methods are simple variations of the Markov chain approach (the birth and death process in continuous time), and illustrate the simplicity of the latter approach.

For other variations of this model or for more complex models the Markov model requires embellishment. Computer simulation then is of value. In particular the model can only be validated for those variations actually tested. Validation of the model as a whole would thus require exhaustive testing of all the alternatives, a monumental task. Furthermore, while all models encounter the problems of parameter estimation, simulation is unique in introducing an explicit statistical variation into the model. In addition, there is some uncertainty as to the selection of appropriate tests and measures for comparison of results.

As a further comparison of simulation, Markovian and traditional statistical models, it is suggested that these alternative techniques be used on one or more variations of the basic personnel flow model and that the results obtained and the process of obtaining the results be examined. This is, in fact, now in progress at Carnegie Tech by one of the authors (Weber). Details will be furnished upon request.

## V. Requirements for Better Models

In developing and testing the kinds of models we have described, a number of things remain to be done. A few of the areas for exploration and experimentation will be suggested here.

Differentiating personnel streams. Of the problems with using positions, ranks, or other kinds of job categories as a main building block in personnel inventory and flow models one of the most important is unequal distribution of chances for promotion or attrition among the persons in a particular rank or job. One way to improve the models (e.g. those suggested in Part III above) is to recognize that there are often different groups to which individuals can be assigned. For example, despite the thorough screening that personnel review boards in the armed services are supposed to give to men who <sup>are</sup> at one rank and might be moved to another, the screening is less comprehensive than it might appear.

In the first place, the group is segmented by structural restrictions. Officers within certain specialties such as medicine or engineering may by law or tradition only be considered for promotions within that specialty. Similarly within industrial organizations, if general foremen must almost always be college graduates, the ranks of foremen can be split into those who are, with some substantial probability for promotion, and those who are not, with almost no chances for promotion.

In the second place, there are subgroupings based on personal characteristics and background variables, either as they are known directly

or as they are reflected in performance or in personal contacts with the men who supply recommendations for promotion. We see direct effects in the sharply reduced promotion opportunities of candidates with certain physical defects, deficiencies in critical prior experience (such as wartime service), or until recently disadvantages of race and sex. We see less direct effects in the relative non-promotability of men who are regarded as unreliable, unsociable, unambitious, and the like.

There is finally an element of geography and personal contact involved. Where the number of candidates is large and the criteria for selection are vague, as is true in the case of many decisions on promotion, it helps if a candidate is "close to" the men making the selections. He is more likely to know that an opportunity is available if it is something to volunteer for; and more likely to be regarded or dismissed strongly as a candidate if the selection committee is making the search.

"Closeness" has several elements. It can include having someone who is in a position to influence the selection decision as a personal sponsor. But more often it is simply a matter of knowing or being known--so that the advantage goes to men from the same current or previous geographical and organizational locales as the men doing the selection.

The grading of candidates by differential probability of promotion or attrition is, then, a matter of law and organizational policy, a matter of organizational tradition about who are and are not good candidates, a matter of the candidate's own personal characteristics and



background, and a matter of the locales in which a man has worked and the contacts he has made.

There are many studies of the influence of small groups of factors-- of education, of personality traits, of family background, of organizational contacts--on selection and participation decisions. But few of these studies have tried to state probabilities that men in a particular subgroup would join, or stay with an organization or would get promoted. And few have provided any sort of insight about how the whole complex of variables interacts to determine the chances for various kinds of individuals in the personnel hierarchy. New hypotheses and new data are needed for the kinds of models we are suggesting.

The end result would be to improve the models by allowing us to disaggregate: to make provision for differences in behavior propensities that matter for the personal and organizational decisions that are reflected in manpower inventory and flow statistics. Disaggregation has another advantage. For in the real world, the chances of some individuals for promotion or for staying with an organization are more stable over time than the chances of others. The existence of subpopulations carries with it an implication that some may have higher priority for promotion than others. While the chances for highranked groups would remain stable and high over time, the chances for other subpopulations would fluctuate sharply according to the number of vacancies which exist. The priorities could shift, also, to reflect the situation in political or semi-political

organizations, where chances for retention or advancement depend on being in the group associated with key power figures.

Promotion and retention in political and quasi-political organizations. In political systems and in quasi-political systems like business, church, and educational organizations, models for at least some levels in the hierarchy need to be adjusted for the effects of "subgroup loyalties" and "patronage" behavior. The nature of these effects is clear, but their magnitudes and the exact conditions which trigger their occurrence are not.

The effect is to define for the organization, or for parts within it, key individuals whose presence and progress affects the presence and progress of others. If they leave, large scale resignations or dismissals of subordinates may occur. If they move up in the organization, the chances for their subordinates also to be promoted greatly increase and the chances of other groups within the system for promotion decrease.

Convergent, parallel, and divergent career lines. Another area for study involves the analysis of career lines within organizational systems. In some systems, like the armed services, there may be a set of parallel and separate career lines, up which a man can move from ensign to admiral, without crossing from one specialty to another. The same situation is approximated in universities where departmental traditions are strong and where it would not be viewed as desirable to make a professor of history a faculty member in economics or political science.

In the typical business organization, however, the trend is toward convergent lines. At lower levels of management there may be dozens of short and separate specialized career hierarchies. As men reach the top of these, they enter a more limited range of lines, each more generalized, each feeding from several of the specialties. At the top of the organization, only a few very generalized kinds of positions are left.

In certain of the professions, though, like medicine the trend may be toward divergent career lines. The young doctor starts, for lack of specialized training and experience, as a general practitioner. Then as he gains the training and experience he becomes a specialist, and as his skills sharpen and his reputation grows, the scope of work that he does may narrow rather than grow. At the end of his career he may do little else than examine eyes, cut out tonsils, or treat heart disease.

In most organizational settings the career lines show a mix of parallel, converging, and diverging characteristics. Learning more about how they differ along such dimensions, though, should be useful in planning models.

## REFERENCES

### General

1. Pigors, P. and Myers, C.A. Personnel Administration, New York: McGraw Hill, 1961, Footnote, page 261.
2. For some worthwhile exceptions see; Cronbach, L.J., and Gleser, G. C. Psychological Tests and Personnel Decisions. Urbana: University of Illinois Press, 1965; King, W.R. Personnel Evaluation and Optimal Assignment. AD 602821, Office of Technical Services, Washington, D.C; Gagne, R. M. (Ed.), Psychological Principles in System Development. New York: Holt, Rinehart and Winston, 1962.
3. Vetter, E.W. "How Firms do Manpower Planning." Nations Business, February, 1964.
4. For example, Losee, J. E. Payfer, G. E. Frohm, W. F. and Eisenberg, B. Methods for Computing Manpower Requirements for Weapons Systems under Development. AD 2 64435; and Demaree, R.G. and Marks, M. R. Development of Qualitative and Quantitative Personnel Requirements Information. MRL-TDR-62-4; Office of Technical Services, Washington, D.C.
5. Packard, K. D. "Probabilistic Forecasting of Manpower Requirements," I.R.E Trans. EM 9, 1962.
6. Fulmer, J. L., Green, R.E., and Hain, P.B. Research Design to Forecast Demand for New Types of Technicians in an Industry. AD 603110, Office of Technical Services, Washington, D.C.
7. Harding, F. D. and Merck, J. W. Markov Chain Theory Applied to the Prediction of Retirement Rates. AD 603110.
8. Merck, J. W. A Mathematical Model of the Personnel Structure of Large Scale Organizations Based on Markov Chains. Unpublished Ph.D. Dissertation, Duke University, 1961.
9. Young, A. and Almond, G. "Predicting Distributions of Staff." The Computer Journal, 3, 1961, pp. 246-250.
10. Ford, F. B. A Technique for the Evaluation of Recruiting Strategy with Fluctuating Availability and Known Demand. AD 294848, Office of Technical Services, Washington, D. C.
11. Gorham, W. "An Application of a Network Flow Model to Personnel Planning," IEEE Transactions, EM 10, 1963.

12. Merck, J. W. and Ford, F. B. Feasibility of a Method of Estimation Short-term and Long-term Effects of Policy Decisions on the Airman Personnel System; WADC Tech. Report 59-38, June 1959, Personnel Lab., Wright Air Dev. Ctr. (reprinted Sept, 1963); ASTIA Doc. No. AD217079.
13. Gaylord, R. H., Farina, A. J. and Spector, P. Operational Analyses of the Naval Personnel System: Part I. Development of a Personnel System Model. AD 2305545. Office of Technical Services, Washington, D.C.
14. Gaylord, R. H., and Knetz, W. J. Operational Analyses of the Naval Personnel System: Part II. Development and Testing of a Machine Simulation of Personnel Operations. AD 262 212, Office of Technical Services, Washington, D.C.
15. Dickman, T. A., Elliot, O. and Hubbard, L. A. Information Retrieval in the Personnel Department, a Survey of Methods used in Scientific and Engineering Organizations. AD 601 488. Office of Technical Services, Washington, D.C.
16. Dunlap, R. A. "Developing an Experience Register for Engineers." IEEE Transactions EM 10, 1963.
17. Marks, M. R. A Data Organization Model for the Personnel Sub-system. AD 266 320, Office of Technical Services, Washington, D. C.
18. Glaser, B. G. "The Impact of Differential Promotion Systems on Careers." IEEE Transactions EM 10, 1963, 21-24.
19. Grusky, O. "The Effects of Succession: A Comparative Study of Military and Business Organizations." In M. Janowitz, The New Military. New York: Russell Sage Foundation, 1965.
20. Kipnis, D. "Mobility expectations and attitudes toward industrial structure," Human Relations, Vol. 70, No. 1, 1964; pp.57-70.
21. Super, D.E. The Psychology of Careers. New York: Harper and Brothers, 1957.
22. Schein, E.H., McKelvey, W.W., Peters, D.R., and Thomas, J.M. "Career Orientations and Perceptions of Rewarded Activity in a Research Organization," Admin. Sci. Quart. Vol. 9, 1965, pp. 333-349.
23. Harding, F. D. and Downey, R. L., Jr. Electronic Engineer Job Types in the Air Force Systems Command. AD 602 447, Office of Technical Services, Washington, D.C.
24. Biderman, A.D. "Sequels to a Military Career: The Retired Military Professional." In M. Janowitz (Ed.). The New Military, New York: Russell Sage Foundation, 1964.

25. Lang, K. "Technology and Career Management in the Military Establishment." In M. Janowitz (Ed.). The New Military. New York: Russell Sage Foundation, 1964.
26. Zald, M. N. and Simon, W. "Career Opportunities and Commitments Among Officers," In M. Janowitz (Ed.). The New Military. New York: Russell Sage Foundation, 1964.

#### Mathematical

27. Crow, E.L. and Weiss, G.H., Mathematical Models for Personnel Promotion, manuscript to be published in Management Science.
28. Seal, H.L., The Mathematics of a Population Composed of k Stationary Strata Each Recruited from the Stratum Below and Supported at the Lowest Level by a Uniform Annual Number of Entrants; BIOMETRIKA, 33, 1945, pp. 226-230.
29. Bartholomew, D.J., A Multi-stage Renewal Process; J. of the Royal Statistical Soc. (B), Vol. 25, No. 2, 1963; pp. 150-168 .
30. Young, A., and Almond, G., Predicting Distributions of Staff, Computer Journal, 3, 1961, pp. 246-250.
31. Vajda, S., The Stratified Semi-stationary Population, BIOMETRIKA, 34, 1947, pp. 243-254.
32. Feller, W., An Introduction to Probability Theory and Its Applications, (2nd Ed.), John Wiley, 1957.
33. Cox, D.R., and Smith, W.L., Queues, Methuen Monograph, John Wiley and Sons, Inc., 1961.

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